COORDINATE TRANSFORMATIONS

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Introduction

- Fundamental activity in land Surveying the integration of multiple sets of data, into a common Geodetic Reference Frame
- In the past sufficient, or even unavoidable ⇨local, **arbitrarily defined** geodetic DATUM
- Satellite positioning and global mapping ⇨***providing products*** in a global geodetic reference frame
- One purpose for a World frame is to ***eliminate use of multiple Geod Datums.***
- Navigation, revision of old maps, cadastral surveying, deformation studies, geo-exploration

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- Problems with a coordinate transformation due to:
  - Distortions and inconsistencies in the local Datum
  - Insufficient knowledge of Geodesy

- Distinction between GRS and GRF ⇨ errors in observations

  ⇨ ***Best estimate*** of transformation parameters
- No unique transformation parameters exist between two GRFs

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- Degree of inconsistency depends on:
  - *Patterns of errors* in the two GRFs and
  - *Choice* of transformation model
- Choice of transformation model :
  - Size of area (sub-network) + distortions
  - Type (3D or 2D) of network + accuracy
- ***3D and 2D transformations*** - ***congruency***

The Models

- ***Full or abridged Molodensky*** formulae (φ, λ, h)

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  translation of origin + ellipsoid parameters changes
- ***Affine transformations*** ⇨ **changes in position, orientation, size and shape of a network**
- ***Conformal or Similarity transformations*** ⇨ unique scale factor

  preserves shape not size
- ***Orthogonal transformations*** ⇨ scale factor unity

3D Transformation models

- Relation between two GRFs requires **6** parameters:
  - *3* parameters for *translation*
  - *3* parameters for *rotations*

  Scale distortion: not part of a transformation

  systematic distortion of positions (network)
- Transformation parameters ⇨ universal / national / local } character
- Few common points ⇨ ***Similarity transformation*** preferable due to simplicity of model

- A similarity transformation smoothes local distortions

  ⇨ division of the area recommended.
- Small values of rotation + scale parameters being expected

  ⇨ ***Bursa-Wolf 3D similarity transformation model***:

$$\begin{bmatrix}X_2\\Y_2\\Z_2\end{bmatrix} = (1+k)\cdot\begin{bmatrix}1 & \varepsilon_z & -\varepsilon_y\\ -\varepsilon_z & 1 & \varepsilon_x\\ \varepsilon_y & -\varepsilon_x & 1\end{bmatrix}\cdot\begin{bmatrix}X_1\\Y_1\\Z_1\end{bmatrix}+\begin{bmatrix}t_x\\t_y\\t_z\end{bmatrix} \quad \text{or} \qquad (2.1)$$

$$\vec{X}_2 = (1+k)\cdot \boldsymbol{R}\cdot \vec{X}_1 + \vec{t}_x$$

- *Translations:* (t_x, t_y, t_z), *rotations:* $(\varepsilon_x, \varepsilon_y, \varepsilon_z)$, *scale component:* k [deviation from unity: $(1+k)$, expressed in ppm].

- This model works well if global or national transformation parameters are to be estimated.

- For limited areas rotations + translations are *significantly correlated*
  ⇨ Part of the rotation affects translations
  ⇨ Translation components differ from their *"national" values*

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- Transformation parameters referring to point (X_o, Y_o, Z_o) (often the *centre of mass of the network*)

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + \begin{bmatrix} k & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & k & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & k \end{bmatrix} \cdot \begin{bmatrix} X_1 - X_o \\ Y_1 - Y_o \\ Z_1 - Z_o \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \quad (2.2)$$

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- ***Minimum number of common points required: 3***

2D Transformation models

- Relatively small networks < 100km × 100km

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conversion of (X,Y,Z) → (φ,λ,h) → (x,y) **map projection coordinates** (common reference ellipsoid and map projection)

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- ***2D similarity transformation*** (*Helmert transformation*)
  $(\Delta x_o, \Delta y_o, \theta, K)$ where $K = (1+k)$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Delta x_o \\ \Delta y_o \end{bmatrix} + K \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad (2.3)$$

- ***Linear expression:***

$$x_2 = ax_1 - by_1 + \Delta x_o \quad (2.4)$$
$$y_2 = bx_1 + ay_1 + \Delta y_o$$

where: $a = K\cos\theta$ and $b = K\sin\theta$

the scale parameter: $K = (a^2+b^2)^{1/2}$ and

the rotation: $\theta = \operatorname{atan}(b/a)$

- **Alternative approach**:

- *Estimation of translation in 3D* (t_x, t_y, t_z) →
- *Application* of translation to data set to be transformed (X′,Y′,Z′) →
- *Conversion* of (X′,Y′,Z′) → (φ′,λ′,h′) → (x,y) →
- ***Full 2D similarity transformation*** *due to non coincidence of centers of mass.*

Data

- For ***the common points two sets of coordinates*** available in all cases:
  GPS data for monitoring deformations expressed in ***ITRF2000***, and coordinates expressed in the *H*ellenic *G*eodetic *R*eference *S*ystem (***HGRS 87***)

1. Simulated network
2. Two networks of 100km × 100km ⟨ Gulf of Corinth / Euboea
3. Large network (250km × 150km)
4. Small network (10km × 10km)

- *Corinth network*: GPS observations at epoch *1995.8*
- *Euboea network*: GPS observations at epoch *1997.7*
- For both networks : Old data (HGRS87) around *1970* → An almost 30 years time interval

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- *Comparing data between epochs for monitoring deformation* ⇨ difficult to distinguish discrepancies due to ⟨ **non coincidence of reference frames** / **real displacements**.

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- ***Simulated network***: A pseudo "HGRS87" coordinate set was created submitting an ITRF2000 GPS data set to a specific transformation and applying random noise.
- ***Table 1.***

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Procedure - Analysis

- For the 3D similarity transformation both expressions were used. Slide 7
- Submatrices ***Ai*** of the design matrix ***A*** are of the form:

$$A_i = \begin{pmatrix} X_i & 0 & -Z_i & Y_i & 1 & 0 & 0 \\ Y_i & Z_i & 0 & -X_i & 0 & 1 & 0 \\ Z_i & -Y_i & X_i & 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.5)$$

or:

$$A_i = \begin{pmatrix} (X - Xo)_i & 0 & (-Z + Zo)_i & (Y - Yo)_i & 1 & 0 & 0 \\ (Y - Yo)_i & (Z - Zo)_i & 0 & (-X + Xo)_i & 0 & 1 & 0 \\ (Z - Zo)_i & (-Y + Yo)_i & (X - Xo)_i & 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.6)$$

- While the ***vector of unknowns*** and the right hand vector are respectively:

$$\hat{x} = \begin{pmatrix} k & \varepsilon_x & \varepsilon_y & \varepsilon_z & t_X & t_Y & t_Z \end{pmatrix}^T$$

$$l_i = \begin{pmatrix} (X_2 - X_1)_i & (Y_2 - Y_1)_i & (Z_2 - Z_1)_i \end{pmatrix}^T$$

- A marked *difference in magnitude exists between the coefficients of the unknowns* ⇨ this affects the compatibility of the significant digits in the *elements of the normal equations matrix* **N**.

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- To overcome this a *two step approach* may be followed:
  1. 3D Translation ◷ application to HGRS87 coordinate sets ◷
  2. estimation of (ε_X ,ε_Y ,ε_Z) and ***k***

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- ***stable LS solution***
- ***no need for iteration*** (very small parameters)
- ***Figure 1, Table 1***

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	Parameters	Networks				
		Simulated	Corinth	Euboea	Large	Small
3D Solutions	$\Delta Xo \pm \sigma_X$(m)	-201.440±.003	-201.444±.208	-199.959±.054	-200.609 ±.161	-200.744±.276
	$\Delta Yo \pm \sigma_Y$(m)	74.270±.003	74.260 ±.208	74.842 ±.054	74.587 ± .161	74.520± .276
	$\Delta Zo \pm \sigma_Z$(m)	245.418±.003	245.413±.208	246.214±.054	245.863±.161	245.544± .276
	$k \pm \sigma_k$(ppm)	0.0±0.0	0.0±0.03	0.0±0.0	0.0±0.02	0.0±0.02
	$\varepsilon_x \pm \sigma\varepsilon_x$(″)	0.0±0.02	0.94±1.2	0.59±0.13	0.61±0.40	2.18± 7.6
	$\varepsilon_y \pm \sigma\varepsilon_y$(″)	0.0±0.01	0.39±0.49	0.26±0.05	0.26±0.17	0.89 ± 3.11
	$\varepsilon_z \pm \sigma\varepsilon_z$(″)	0.0±0.02	0.80±1.02	0.51±0.11	0.52±0.34	1.84 ± 6.45
2D Solutions	$\Delta x_o \pm \sigma_X$(m)	148.729±.940	120.185±15.869	132.694±3.152	131.992±3.456	89.273± 2.432
	$\Delta y_o\ \sigma_Y$(m)	309.340±.940	292.535±15.869	303.605±3.152	309.750±3.456	322.510± 2.432
	$k \pm \sigma_k$(ppm)	5.0±0.22	0.51±3.7	3.4±0.74	4.8±0.81	7.1±2.9
	$\omega \pm \sigma_\omega$(″)	-0.17±0.05	-1.48±0.77	-0.91±0.15	-0.97±0.17	3.10±.61
Alternative Approach	$\Delta x_o \pm \sigma_X$(m)			-17.853±3.162		-60.127±14.294
	$\Delta y_o\ \sigma_Y$(m)			-6.348±3.162		18.590±14.294
	$k \pm \sigma_k$(ppm)			1.9±1.05		4.3±3.4
	$\omega \pm \sigma_\omega$(″)			4.0±1.05		-14.5±3.4

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Table 1 Transformation parameters for 3D and 2D models with respective r.m.s.

Types of solutions	Range of discrepancies in cm				
	Simulated Network	*Corinth*	*Euboea*	*Large Network*	*Small Network*
3D solution in two steps (**case 1**)	1-2.5	3-115	1-33	3.5-170	1-155
2D solution (**case 2**)	1-2.5	1-34	1-21	1-65	0-6.5
⇨ **3D** solution projected to **2D** (**case 3**)	1-2	5-48	2-27	2-60	1-19
⇨ Comparison **3D – 2D** solution (**case 4**)	1-2	2.5-29	1-28	1-40	1-13
2D solution after **3D** translation (**Alternative Approach**)			1-21		1-8

Table 2 Range of discrepancies in cm, for all networks and all types of transformation models.

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Discussion - Conclusions

- In the case of the **simulated network** all discrepancies in 3D + 2D solutions ***< 2-3cm*** of the same order as LS residuals

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- ***Discrepancies only due to random errors***
- In all **other cases** *discrepancies* and *residuals are significant* (***several tenths of cm***).

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- Due to ***the existence of a displacement field*** both in the Corinthian gulf and the vicinity of Euboea.

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- In the case of the **small network** :
  discrepancies in the 3D > than in 2D ◷
  in agreement with the generally accepted concept that
  2D transformations are preferable for small networks.
- If **local parameters** are to be estimated ◷
  it may be irrelevant whether 3D or 2D is used even for large networks.
- In the case of **3D transformation** ◷
  preferable ***the two steps approach*** ⇨ LS solution more stable; no iterations
- For **monitoring displacements** ◷
  The ***choice of the appropriate transformation*** (2D or 3D or any combination) is ***not easily answered***.

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